



# Directed Percolation Process Advected by Compressible Velocity Field

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## Abstract

The directed percolation process in the vicinity of non-equilibrium phase transition is studied by the means of field theoretic methods. It will be assumed that percolation takes place in a compressible environment, which will be generated by the modified Kraichnan model. We will discuss differences with the incompressible case and describe how the given model can be renormalized. The renormalization constants will be given to the one-loop order.

## 1 Introduction

Nearly all physical phenomena are due to non-equilibrium processes. Examples include chemical reactions in organisms, heat transport in the atmosphere, social-economical systems and so on. Despite the considerable effort the general principles for far-from-equilibrium statistical mechanics are still missing. Although general study of such systems is very demanding, there exist situations, which are amenable to theoretical analysis. Analogously to the equilibrium critical behavior, in the vicinity of some special points the scale invariance emerges that can be handled within renormalization group (RG) technique. The famous prototype is the directed percolation (DP) process [1], which could be thought either as a kind of epidemic process[2] or as Reggeon field theory in particle physics[3]. The phase transition [2] in these systems occurs between the absorbing state (no infected individuals) and active state (nonzero number of infected individuals).

The deviations from the ideality are known to have a profound impact. Immunization, long-range interactions or quenched disorder change the critical behavior and in general give rise to new universality class [4, 5].

The aim of this paper is to study the directed percolation process in the presence of compressible advective environment. Environment will be modeled by the modified Kraichnan model [6, 7], which allows us to analyze effect of finite correlation time of velocity fluctuations and of compressibility itself.

## 2 The model

The continuum description of DP in terms of a density  $\psi = \psi(t, \mathbf{x})$  of infected individuals typically arises from a coarse-graining procedure in which a large number of microscopic degrees of freedom were averaged out. Their reminiscence is the presence of a Gaussian noise in the corresponding Langevin equation. The mathematical model must obviously respect the absorbing state condition:  $\psi = 0$  is always a stationary state. The coarse grained stochastic equation then reads [8]

$$\partial_t \psi = D_0(\nabla^2 - \tau_0)\psi - \frac{g_0 D_0}{2} \psi^2 + \eta, \quad (1)$$

where  $\eta$  denotes the noise term,  $\partial_t = \partial/\partial t$  is the time derivative,  $\nabla^2$  is the Laplace operator,  $D_0$  is the diffusion constant,  $g_0$  is the coupling constant and  $\tau_0$  measures deviation from the threshold value for injected probability. The latter quantity can be thought as an analog to the temperature variable in the standard  $\varphi^4$ -theory. [8, 9]. Here and henceforth we distinguish unrenormalized (with subscript “0”) quantities from renormalized ones (without subscript “0”). The renormalized fields will be later denoted with the subscript  $R$ .

Although the rigorous proof is lacking, it is generally believed that Langevin equation (1) captures the gross properties of the percolation process and should contain relevant information about the large-scale behavior of the non-equilibrium phase transition between active  $\psi > 0$  and absorbing state  $\psi = 0$ . Due to the absorbing state condition the correlator of  $\eta$  has the following form

$$\langle \eta(t_1, \mathbf{x}_1) \eta(t_2, \mathbf{x}_2) \rangle = g_0 D_0 \psi(t_1, \mathbf{x}_1) \delta(t_1 - t_2) \delta^{(d)}(\mathbf{x}_1 - \mathbf{x}_2). \quad (2)$$

The next step consists in the incorporation of the velocity fluctuations. The standard way based on the replacement  $\partial_t$  by the Lagrangian derivative  $\partial_t + (\mathbf{v} \cdot \nabla)$  is not sufficient. As was shown in [10] the presence of compressibility requires the following replacement

$$\partial_t \rightarrow \partial_t + (\mathbf{v} \cdot \nabla) + a_0(\nabla \cdot \mathbf{v}), \quad (3)$$

where  $a_0$  is an additional parameter. From the point of view of RG the introduction of  $a_0$  is necessary, since it ensures multiplicative renormalizability of the model.

Our main aim here is to show how the influence of compressibility together with finite time correlated velocity field could be incorporated into the model of DP. To this end we employ the model based on the modified Kraichnan model [6, 11]. Such model properly describes main features of the turbulent advection-diffusion propagation.

Following the work [6] we consider the velocity field to be random Gaussian variable with zero mean and translational invariant correlator chosen in the form

$$\langle v_i(t, \mathbf{x}) v_j(0, \mathbf{0}) \rangle = \int \frac{d\omega}{2\pi} \int \frac{d^d \mathbf{k}}{(2\pi)^d} D_v(\omega, \mathbf{k}) e^{-i\omega t + \mathbf{k} \cdot \mathbf{x}}, \quad (4)$$

where the kernel function  $D_v(\omega, \mathbf{k})$  is given as

$$D_v(\omega, \mathbf{k}) = [P_{ij}^k + \alpha Q_{ij}^k] \frac{g_{10} u_{10} D_0^3 k^{4-d-y-\eta}}{\omega^2 + u_{10}^2 D_0^2 (k^2 - \eta)^2}. \quad (5)$$

Here  $P_{ij}^k = \delta_{ij} - k_i k_j / k^2$  is transverse and  $Q_{ij}^k$  longitudinal projection operator,  $k = |\mathbf{k}|$ , and  $d$  is the dimensionality of the  $\mathbf{x}$  space. Positive parameter  $\alpha > 0$  can be interpreted as a deviation from the incompressibility condition  $\nabla \cdot \mathbf{v} = 0$ . The case  $\alpha = 0$  was studied previously [10, 12, 13]. The coupling constant  $g_{10}$  and exponent  $y$  describe the equal-time velocity correlator or, equivalently, the energy spectrum [14, 15]. On the other hand, the constant  $u_{10} > 0$  and the exponent  $\eta$  describe the characteristic frequency of the mode  $k$ .

The exponents  $y$  and  $\eta$  are analogous to the standard expansion parameter  $\varepsilon = 4 - d$  in the static critical phenomena. Later on it will be shown (Sec. 3.1) that the upper critical dimension of the pure percolation problem is indeed  $d_c = 4$ . Therefore we retain the standard notation for the exponent  $\varepsilon$ . According to the general rules [9, 15] of RG approach we formally assume that the exponents  $\varepsilon, y$  and  $\eta$  are of the same order of magnitude and in principle they constitute small expansion parameters in a perturbation sense.

The general form of kernel function in (5) contains various special limits. Their analysis is usually simpler and thus allows us to gain deeper physical insight. The possible limits are

- i) Rapid-change model, which is obtained in the limit  $u_{10} \rightarrow \infty$ ,  $g'_{10} \equiv g_{10}/u_{10} = \text{const}$ . For this case we have  $D_v(\omega, \mathbf{k}) \propto g'_{10} D_0 k^{-d-y+\eta}$ , and therefore the velocity field is now  $\delta$ -correlated in time variable.
- ii) 'Frozen' velocity field: given by the limit  $u_{10} \rightarrow 0$  and corresponds to the following form of Kernel function  $D_v(\omega, \mathbf{k}) \propto g_0 D_0^2 \pi \delta(\omega) k^{2-d-y}$ , In other words, the velocity field is quenched (time-independent).
- iii) Pure potential velocity field:  $\alpha \rightarrow \infty$  with  $\alpha g_{10} = \text{constant}$ . This limit is similar to the model of random walks in random environment with long-range correlations [16].
- iv) Turbulent advection obtained setting  $y = 8/3$  and  $\eta = 4/3$ . This choice leads directly to the famous Kolmogorov "five-thirds" law for the spatial velocity correlations [14].

For the effective use of RG method it is advantageous to reformulate the stochastic problem (1-5) into the field-theoretic language. This can be achieved in the standard fashion [15, 17, 18] and the resulting dynamic functional can be written as a sum of three terms

$$\mathcal{S}[\varphi] = \mathcal{S}_{\text{diff}}[\varphi] + \mathcal{S}_{\text{vel}}[\varphi] + \mathcal{S}_{\text{int}}[\varphi], \quad (6)$$

where  $\varphi = \{\tilde{\psi}, \psi, \mathbf{v}\}$  is the complete set of fields. In what follows  $\psi^\dagger$  is the response field appearing after the noise field was integrated out [15]. The first term represents nothing else as free part of the equation (1) and can be written as follows

$$\mathcal{S}_{\text{diff}}[\varphi] = \tilde{\psi}[\partial_t - D_0 \nabla^2 + D_0 \tau_0] \psi, \quad (7)$$

where the necessary integrations over time-spatial variables were omitted. For example second term stands for

$$\tilde{\psi} \nabla^2 \psi = \int dt \int d^d \mathbf{x} \left\{ \tilde{\psi}(t, \mathbf{x}) \nabla^2 \psi(t, \mathbf{x}) \right\}. \quad (8)$$

Since velocity fluctuations are considered as Gaussian stochastic variables, the averaging procedure for them corresponds to the functional integration with quadratic functional

$$\mathcal{S}_{\text{vel}}[\mathbf{v}] = -\frac{1}{2} \mathbf{v}_i D_{ij}^{-1} \mathbf{v}_j, \quad (9)$$

where  $D_{ij}^{-1}$  is the kernel of the inverse linear operation in (4) and summation over repeated indices is implied. The interaction part can be written as

$$\mathcal{S}_{\text{int}}[\varphi] = \tilde{\psi} \left\{ \frac{D_0 \lambda_0}{2} [\psi - \tilde{\psi}] - \frac{u_{20}}{2D_0} \mathbf{v}^2 + (\mathbf{v} \cdot \nabla) + a_0 (\nabla \cdot \mathbf{v}) \right\} \psi. \quad (10)$$

All but the third term in (10) stems directly from the nonlinear parts in eqs. (1) and (3). The third term proportional to  $\propto \psi \psi \mathbf{v}^2$  deserves a special consideration. Presence of such term is prohibited in the original Kraichnan model due to the underlying Galilean invariance. However in our case the general form of the velocity kernel function doesn't lead to such restriction. Also by direct inspection of the perturbative RG method, one can show that such term will indeed be generated (Consider first three Feynman graphs in the expansion (24)). Such term was also introduced in our previous work [13], where the incompressible analog was studied. It was shown that such term doesn't lead to the significant differences with respect to the quantitative values of universal quantities.

Basic ingredients of stochastic theory, correlation and response functions of the concentration field  $\psi(t, \mathbf{x})$ , can be in principle computed as functional averages with respect to the weight functional  $\exp(-\mathcal{S})$ . Further the field-theoretic formulation summarized in (7)-(10) has the additional advantage, namely it is amenable to the full machinery of (quantum) field theory [9, 15]. In subsequent section we will apply RG perturbative technique [15] that allows us to study the model in the vicinity of its upper critical dimension  $d_c = 4$ .

### 3 Renormalization group analysis

The important goal of statistical theories is the determination of correlation and response functions (usually called Green functions) of the dynamical fields as functions of the space-time coordinates. Graphically one can represent these functions

$Q$	$\psi, \tilde{\psi}$	$\mathbf{v}$	$D_0$	$\tau_0$	$g_{10}$	$\lambda_0$	$u_{10}$	$u_{20}, a_0, \alpha$
$d_Q^k$	$d/2$	-1	-2	2	$y$	$\varepsilon/2$	$\eta$	0
$d_Q^\omega$	0	1	1	0	0	0	0	0
$d_Q$	$d/2$	1	0	2	$y$	$\varepsilon/2$	$\eta$	0

Table 1: Canonical dimensions of the fields and parameters for the model (7)-(10).

in a form of sums over Feynman diagrams [9]. Near criticality  $\tau = 0$  large fluctuations on all spatio-temporal scales dominate the behavior of the system, which in turn results into the divergences in Feynman graphs. The RG technique allows us to deal with them and at the same time it serves us as an efficient technique for determination of possible large scale behavior. Also it provides us with the perturbative algorithm for estimation of universal quantities in the form of formal expansion around upper critical dimension.

The critical behavior of the model is analyzed using standard RG procedure [15]. This procedure requires action to be multiplicatively renormalizable and this goal can be achieved by adding a new term  $\psi^\dagger \psi v v$  into the total action with new independent parameter (charge)  $u_2$ .

### 3.1 Canonical dimensions

The analysis of ultraviolet (UV) divergences is based on the power counting procedure. This step allows us to identify UV divergent structures in the perturbation theory. For translational invariant systems it is sufficient to analyze only 1-particle irreducible (1PI) graphs [9, 15].

In contrast to the static models, dynamic models contain two independent scales: frequency scale and momentum scale. The corresponding dimensions for each quantity can be found using the standard normalization conditions

$$\begin{aligned} d_k^k &= -d_x^k = 1, & d_\omega^k &= d_t^k = 0, \\ d_k^\omega &= d_x^\omega = 0, & d_\omega^\omega &= -d_t^\omega = 1 \end{aligned} \quad (11)$$

together with condition to field-theoretic action (6) be a dimensionless quantity. Further based on the values  $d_Q^\omega$  and  $d_Q^k$ , one can introduce the total canonical dimension  $d_Q$

$$d_Q = d_Q^k + 2d_Q^\omega, \quad (12)$$

whose form can be obtained from the comparison of IR most relevant terms ( $\partial_t \propto \nabla^2$ ) in the action.

The dimensions of all quantities for the model are summarized in Table 1. It follows that the model is logarithmic (when coupling constants are dimensionless) at  $\varepsilon = y = \eta = 0$ , and the UV divergences have the form of the poles in these small parameters. The total dimension  $d_Q$  plays for the dynamical models the same role as does the conventional (momentum) dimension in static problems. The total canonical dimension of an arbitrary 1PI Green function is given by the relation

$$d_\Gamma = d_\Gamma^k + 2d_\Gamma^\omega = d + 2 - \sum_i N_i d_i, \quad i \in \{\tilde{\psi}, \psi, \mathbf{v}\} \quad (13)$$

$\Gamma_{1\text{PI}}$	$\Gamma_{\psi, \tilde{\psi}}$	$\Gamma_{\tilde{\psi}\psi\mathbf{v}}$	$\Gamma_{\tilde{\psi}^2\psi}$	$\Gamma_{\tilde{\psi}\psi^2}$	$\Gamma_{\tilde{\psi}\psi\mathbf{v}^2}$
$d_\Gamma$	2	1	$\varepsilon$	$\varepsilon$	0

Table 2: Canonical dimensions of the potentially divergent 1PI functions.

Using this relation one immediately obtains the divergent functions listed in Tab. 2.

### 3.2 Computation of the RG constants

Here, we will summarize the main steps of the perturbative aspects of the RG approach. Perturbation theory is developed starting from the free part of the action, which are graphically represented as lines in the Feynman diagrams. On the other hand the non-linear terms in (10) play the role of vertices.

For the calculation of RG constants we have used dimensional regularization in the minimal subtraction scheme (MS). In this scheme the expansion parameters are  $\varepsilon$ ,  $y$  and  $\eta$  and the poles in renormalization constants can be realized in the form of their linear combination. Because the finite correlated case involves two different dispersion laws:  $\omega \propto k^2$  for the scalar and  $\omega \propto k^{2-\eta}$  for the velocity fields, the calculations for the renormalization constants become rather cumbersome already in the one-loop approximation [6, 11]. However, as was shown in [19] to the two-loop order it is sufficient to consider  $\eta = 0$ . This choice substantially simplifies the practical calculations and as can be seen from the explicit expressions (25), the only poles to the one-loop order are  $1/\varepsilon$  and  $1/y$ . We stress that this simple picture pertains only to the lowest orders in perturbation scheme. In higher order terms poles in the form of general linear combinations in  $\varepsilon$ ,  $\eta$  and  $y$  are expected to arise.

The perturbation theory of the model (6) is amenable to the standard Feynman diagrammatic expansion [9, 15]. Studied model contains three different types of propagators and they are easily read off from the Gaussian part of the model given by (7) and (9), respectively. Their graphical representation is depicted in Fig. 1. The corresponding algebraic expressions are

$$\langle \psi \tilde{\psi} \rangle_0 = \langle \tilde{\psi} \psi \rangle_0^* = \frac{1}{-i\omega + D(k^2 + \tau)}, \quad (14)$$

$$\langle \mathbf{v} \mathbf{v} \rangle_0 = [P_{ij}^k + \alpha Q_{ij}^k] \frac{g_{10} u_{10} D_0^3 k^{4-d-y-\eta}}{\omega^2 + u_{10}^2 D_0^2 (k^{2-\eta})^2}. \quad (15)$$

The interaction vertices from the nonlinear part of the action (10) describe the fluctuation effects connected with the percolation process itself, advection of concentration field and the interactions between the velocity components, respectively. With every such vertex the so-called vertex factor [15]

$$V_N(x_1, \dots, x_N; \varphi) = \frac{\delta^N \mathcal{S}_{\text{int}}[\varphi]}{\delta \varphi(x_1) \dots \delta \varphi(x_N)}, \quad \varphi \in \{\tilde{\psi}, \psi, \mathbf{v}\}$$

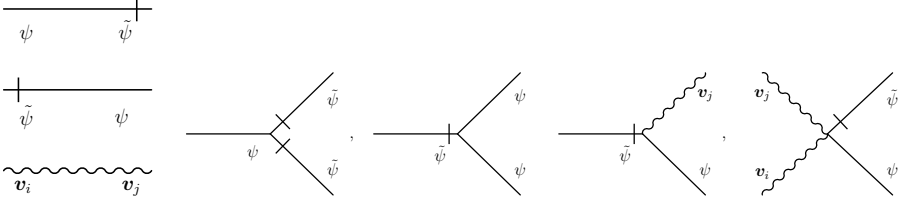


Figure 1: Diagrammatic representation of the elements of perturbation theory.

is associated. In our model we have to deal with four interaction vertices, which are also graphically depicted in the Fig. 1. The corresponding vertex factors are

$$V_{\tilde{\psi}\psi\psi} = -V_{\tilde{\psi}\tilde{\psi}\psi} = D_0\lambda_0, \quad V_{\tilde{\psi}\psi\mathbf{v}} = -\frac{u_{20}}{D_0}, \quad V_{\tilde{\psi}\psi\mathbf{v}\mathbf{v}} = ik_i + ia_0q_i, \quad (16)$$

where in the last terms  $k_i$  is the momentum of the field  $\psi$  and  $q_i$  is the momentum carried by the velocity field  $\mathbf{v}$ . The presence of the interaction vertex  $V_{\tilde{\psi}\psi\mathbf{v}\mathbf{v}}$  leads to the proliferation of the new Feynman graphs, which were absent in the previous studies [10, 12, 13]. By direct inspection of the Feynman diagrams one can easily observe that the real expansion parameter is rather  $\lambda_0^2$  than  $\lambda_0$ . This is a consequence of the duality symmetry of the action for the pure percolation problem with respect to time reversal  $\psi(t, \mathbf{x}) \rightarrow -\tilde{\psi}(-t, \mathbf{x})$ ,  $\tilde{\psi}(t, \mathbf{x}) \rightarrow -\psi(-t, \mathbf{x})$ . Therefore it is advantageous to introduce new charge  $g_{20}$  by as  $g_{20} = \lambda_0^2$ . We note that in the presence of compressible velocity field this transformation has to be supplemented by additional transformation  $a_0 \rightarrow 1 - a_0$  as can be directly seen. Using Tab. 1 we introduce the renormalized parameters via

$$\begin{aligned} D_0 &= DZ_D, & \tau_0 &= \tau Z_\tau + \tau_c, & a_0 &= aZ_a, \\ g_{10} &= g_1\mu^{y+\eta}Z_{g_1}, & u_{10} &= u_1\mu^\eta Z_{u_1}, & \lambda_0 &= \lambda\mu^\varepsilon Z_\lambda, \\ g_{20} &= g_2\mu^{2\varepsilon}Z_{g_2}, & u_{20} &= u_2Z_{u_2}, \end{aligned} \quad (17)$$

where  $\mu$  is the reference mass scale in the MS scheme [9] and  $\tau_c$  stands for fluctuation shift of the critical point. The choice (17) together with renormalization of fields  $\tilde{\psi} = Z_{\tilde{\psi}}\tilde{\psi}_R$ ,  $\psi = Z_\psi\psi_R$ ,  $\mathbf{v} = Z_v\mathbf{v}_R$  is sufficient to have fully UV renormalized theory. The total renormalized action for the renormalized fields  $\varphi_R \equiv \{\tilde{\psi}_R, \psi_R, \mathbf{v}_R\}$  can be written in the compact form

$$\begin{aligned} S_R[\varphi_R] &= \tilde{\psi}_R \left[ Z_1 \partial_t - Z_2 D \nabla^2 + Z_3 D \tau + Z_4 (\mathbf{v}_R \cdot \nabla) + a Z_5 (\nabla \cdot \mathbf{v}_R) \right] \psi_R \\ &- \frac{D\lambda}{2} [Z_6 \tilde{\psi}_R - Z_7 \psi_R] \tilde{\psi}_R \psi_R - Z_8 \frac{u_2}{2D} \tilde{\psi}_R \psi_R \mathbf{v}_R^2 + \frac{\mathbf{v}_R D_{Rv}^{-1} \mathbf{v}_R}{2}. \end{aligned} \quad (18)$$

The relations between renormalization constants can be directly read off from the action (18), which yields

$$\begin{aligned}
 Z_1 &= Z_\psi Z_{\psi^\dagger}, & Z_2 &= Z_\psi Z_{\psi^\dagger} Z_D, \\
 Z_3 &= Z_\psi Z_{\psi^\dagger} Z_D Z_\tau, & Z_4 &= Z_\psi Z_{\psi^\dagger} Z_v, \\
 Z_5 &= Z_\psi Z_{\psi^\dagger} Z_v Z_a, & Z_6 &= Z_\psi Z_{\psi^\dagger}^2 Z_D Z_\tau, \\
 Z_7 &= Z_\psi^2 Z_{\psi^\dagger}^2 Z_D Z_\lambda, & Z_8 &= Z_\psi Z_{\psi^\dagger} Z_v^2 Z_{u_2} Z_D^{-1}.
 \end{aligned} \tag{19}$$

After the theory is made UV finite through the appropriate choice of RG constants  $Z_1, \dots, Z_8$  one can use the relations (19) to obtain corresponding RG constants for the fields and parameters appearing in (17). Note that according to the general rules of RG the nonlocal terms should not be renormalized. From the inspection of kernel function (5) one thus obtain two additional relations  $1 = Z_{u_1} Z_D$  and  $1 = Z_{u_1} Z_{g_1} Z_D^3 Z_v^{-2}$ .

Now we give a brief overview of how the renormalization constants were computed. The number of divergent Feynman graphs do not pose a serious problem to the first order of the perturbation theory. Moreover their analysis is to some extent simplified by the two facts:

1. Integral of a power of internal momenta is zero in dimensional regularization. Hence the tadpole diagrams are immediately discarded.
2. Closed circuits of propagators  $\tilde{\psi}\psi$  vanish identically, which is a consequence of the Itô time discretization[15] that we have here considered.

From the two-point Green functions only  $\Gamma_{\tilde{\psi}\psi}$  deserves special attention. For it one can write down the following Dyson equation

$$\Gamma_{\tilde{\psi}\psi} = i\omega Z_1 - Dp_2 Z_2 - D\tau Z_3 + \text{[tadpole diagram]} + \frac{1}{2} \text{[tadpole diagram]}. \tag{20}$$

The perturbation expansion for the interaction vertices can be successively presented in the following way:

$$\begin{aligned}
 \langle \tilde{\psi}\psi \mathbf{v} \rangle_{1PI} \Gamma_{\tilde{\psi}\psi \mathbf{v}} &= -ip_j Z_4 - iaq_j Z_5 + \text{[triangle diagram]} + \frac{1}{2} \text{[triangle diagram]} \\
 &+ \text{[tadpole diagram]} + \text{[tadpole diagram]}, \tag{21}
 \end{aligned}$$

$$\Gamma_{\tilde{\psi}\tilde{\psi}\psi} = D\lambda Z_6 + \text{[triangle diagram]} + \frac{1}{2} \text{[triangle diagram]} + \text{[triangle diagram]}, \tag{22}$$



$$\Gamma_{\tilde{\psi}\psi\psi} = -D\lambda Z_7 + \text{triangle diagram with two wavy lines} + \frac{1}{2} \text{triangle diagram with one wavy line} + \text{triangle diagram with three wavy lines}, \quad (23)$$

$$\begin{aligned} \Gamma_{\tilde{\psi}\psi\mathbf{v}\mathbf{v}} = & \frac{u_2}{D}\delta_{ij}Z_8 + \text{square diagram with two wavy lines} + \text{square diagram with one wavy line} + \frac{1}{2} \text{square diagram with no wavy lines} + \text{triangle diagram with two wavy lines} \\ & + \text{triangle diagram with one wavy line} + \text{triangle diagram with no wavy lines} + \text{triangle diagram with two wavy lines} \end{aligned} \quad (24)$$

For the graphs whose symmetry coefficient [15] is different from 1, we have explicitly stated its numerical value. Note again that in the language of Feynman graphs the need for the term  $\propto \tilde{\psi}\psi\mathbf{v}^2$  can be traced out to the first three Feynman graphs in (24), which in the sum doesn't cancel out as is the case for incompressible velocity field.

The computation of the diverging parts of the Feynman graphs follows the usual methods of dimensional regularization [15] and the 1-loop results are

$$\begin{aligned} Z_1 &= 1 + \frac{g_1\alpha a(1-a)}{(1+u_1)^2 y} + \frac{g_2}{4\epsilon}, \\ Z_2 &= 1 - \frac{g_1}{4(1+u_1)y} \left[ 3 + \alpha \left( \frac{u_1-1}{u_1+1} - \frac{4a(1-a)u_1}{(1+u_1)^2} \right) \right] + \frac{g_2}{8\epsilon}, \\ Z_3 &= 1 + \frac{g_1\alpha a(1-a)}{(1+u_1)^2 y} + \frac{g_2}{2\epsilon}, \\ Z_4 &= 1 + \frac{g_1}{4(1+u_1)^2 y} \left[ \alpha \left( 1 + \frac{4a(1-a)u_1}{1+u_1} \right) - u_2(6+6u_1+2\alpha u_1) \right] + \frac{g_2}{4\epsilon}, \\ Z_5 &= 1 + \frac{g_1\alpha}{4(1+u_1)^2 y} \left[ 1 + 2(1-a) \left( 2a - \frac{1}{1+u_1} \right) \right] \\ &\quad - \frac{g_1 u_2}{4a(1+u_1)y} \left[ 3 + \alpha - \frac{2\alpha(1-a)}{1+u_1} \right] + \frac{g_2(4a-1)}{8a\epsilon}, \\ Z_6 &= 1 - \frac{g_1\alpha(1-a)}{(1+u_1)y} \left[ 1 - a - \frac{2a}{1+u_1} \right] + \frac{g_2}{\epsilon}, \\ Z_7 &= 1 - \frac{g_1\alpha a}{(1+u_1)y} \left[ a - \frac{2(1-a)}{1+u_1} \right] + \frac{g_2}{\epsilon}, \\ Z_8 &= 1 + \frac{g_1}{2(1+u_1)y} \left[ \alpha \frac{2a(1-a)+1}{1+u_1} - \frac{\alpha a(1-a)}{u_2(1+u_1)^2} - u_2(3+\alpha) \right] + \frac{g_2}{2\epsilon}. \end{aligned} \quad (25)$$

The ubiquitous geometric factors stemming from the angular integration were included into the renormalized charges  $g_1$  and  $g_2$  via the redefinitions:  $g_1/16\pi^2 \rightarrow g_1, g_2/16\pi^2 \rightarrow g_2$ . The equations (25) must satisfy certain conditions dictated

by the aforementioned time reversal symmetry. This symmetry results into the following conditions [10] for the renormalization constants

$$\begin{aligned} Z_i(a) &= Z_i(1-a) \quad i \in \{1, 2, 3, 4, 8\}, \\ Z_6(a) &= Z_7(1-a), \quad Z_7(a) = Z_6(1-a), \\ Z_1(a) - aZ_5(a) &= (1-a)Z_5(1-a), \end{aligned} \quad (26)$$

where the RG constants are considered as functions of renormalized parameter  $a$ . These relations can be directly checked for our results (25). Further the relations (19) could be inverted with respect to the RG constants for the fields and parameters in a straightforward manner to yield

$$\begin{aligned} Z_D &= Z_2 Z_1^{-1}, & Z_\tau &= Z_3 Z_2^{-1}, & Z_v &= Z_4 Z_1^{-1}, \\ Z_a &= Z_5 Z_4^{-1}, & Z_\psi &= Z_1^{1/2} Z_6^{-1/2} Z_7^{1/2}, & Z_{\psi^\dagger} &= Z_1^{1/2} Z_6^{1/2} Z_7^{-1/2}, \\ Z_{u_1} &= Z_1 Z_2^{-1}, & Z_\lambda &= Z_1^{-1/2} Z_2^{-1} Z_6^{1/2} Z_7^{1/2}, & Z_{g_2} &= Z_1^{-1} Z_2^{-2} Z_6 Z_7, \\ Z_{u_2} &= Z_2 Z_8 Z_4^{-2}, & Z_{g_1} &= Z_2^{-2} Z_4^2. \end{aligned} \quad (27)$$

After insertion explicit results for renormalization constants (25) one obtains desired RG constants.

## 4 Fixed points and scaling regimes

The scaling behavior in the infrared (IR) limit can be studied by analyzing RG flow as  $\mu \rightarrow 0$  after the renormalization procedure to given order of perturbation scheme is performed. The possible IR asymptotic behavior is governed by the fixed points (FPs) of the  $\beta$ -functions [15]. The fixed points  $g^* = \{g_1^*, g_2^*, u_1^*, u_2^*, a^*\}$  can be found from requirement that all  $\beta$  functions simultaneously vanish

$$\beta_{g_1}(g^*) = \beta_{g_2}(g^*) = \beta_{u_1}(g^*) = \beta_{u_2}(g^*) = \beta_a(g^*) = 0, \quad (28)$$

where the  $\beta$  functions, which express the flows of parameters under the RG transformation [9, 15], are defined as  $\beta_g = \mu \partial_\mu g|_0$ . Here  $\dots|_0$  denotes fixed bare parameters. Whether the given FP could be realized in physical systems (IR stable) or not (IR unstable) is determined by the eigenvalues of the matrix  $\Omega = \{\Omega_{ij}\}$  with the components  $\Omega_{ij} = \partial \beta_i / \partial g_j$ , where  $\beta_i$  stands for the full set of  $\beta$  functions and  $g_j$  is the full set of charges  $\{g_1, g_2, u_1, u_2, a\}$ . For the IR stable FP the real part eigenvalues of matrix  $\Omega$  are strictly positive. In general these conditions determine the region of stability for the given FP in terms of  $\varepsilon, \eta$  and  $y$ . Further one exploit fact that the bare Green functions are independent of  $\mu$  to obtain RG equation. Applying the differential operator  $\mu \partial_\mu$  at fixed bare quantities leads to the following equation for the renormalized Green function  $G_R$

$$\{D_{\text{RG}} + N_\psi \gamma_\psi + N_{\psi^\dagger} \gamma_{\psi^\dagger} + N_v \gamma_v\} G_R(e, \mu, \dots) = 0, \quad (29)$$

where  $e$  is the full set of renormalized counterparts of the bare parameters  $e_0 = \{D_0, \tau_0, u_{10}, u_{20}, g_{10}, g_{20}, a_0\}$  and  $\dots$  denotes other parameters, such as spatial or

time variables. The RG operator  $D_{\text{RG}}$  is given in the form

$$D_{\text{RG}} \equiv \mu \partial_\mu|_0 = \mu \partial_\mu + \sum_g \beta_g \partial_g - \gamma_D D_D - \gamma_\tau D_\tau, \quad (30)$$

where  $g \in \{u_1, u_2, g_1, g_2, a\}$ ,  $D_x = x \partial_x$  for any variable  $x$ , and  $\gamma_x$  are anomalous dimensions of the quantity  $x$  and are defined as  $\gamma_x \equiv \mu \partial_\mu \ln Z_x|_0$ . Application of this definition on the relations (17) leads in straightforward manner to the equations

$$\begin{aligned} \beta_{g_1} &= g_1(-y + 2\gamma_D - 2\gamma_v), & \beta_{g_2} &= g_2(-\epsilon - \gamma_{g_2}), & \beta_{u_1} &= u_1(-\eta + \gamma_D), \\ \beta_{u_2} &= -u_2\gamma_{u_2}, & \beta_a &= -a\gamma_a. \end{aligned} \quad (31)$$

Last equation suggests that either  $a = 0$  or  $a \neq 0$  is satisfied. However, as the explicit results (25) show parameter  $a$  could also appear in the denominator. Similar reasoning also applies for the function  $\beta_{u_2}$ .

From the structure of anomalous dimensions it is clear that the resulting systems of equations for FPs is quite involved. Hence in order to gain some physical insight into their structure it would be reasonable to divide to overall analysis into some special cases.

## 5 Conclusions

In this article we have summarized main points of the field-theoretic study of directed percolation process in the presence of compressible velocity field. We have obtained field theoretical action and presented renormalization procedure of the model to the one-loop order. The next step consists in detailed analysis of the possible scaling regimes, which however due to the complicated form of RG functions will be published elsewhere [20].

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